

EXTENSIONS OF VALUATION RINGS IN CENTRAL SIMPLE ALGEBRAS

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ABSTRACT. Certain subrings R of simple algebras Q , finite dimensional over their center K , are studied. These rings are called Q -valuation rings since they share many properties with commutative valuation rings. Let V be a valuation ring of K , the center of Q , and let \mathcal{R} be the set of Q -valuation rings R in Q with $R \cap K = V$, then $|\mathcal{R}| \geq 1$. This extension theorem, which does not hold if one considers only total valuation rings, was proved by N. I. Dubrovin. Here, first a somewhat different proof of this result is given and then information about the set \mathcal{R} is obtained. **Theorem.** *The elements in \mathcal{R} are conjugate if V has finite rank.* **Theorem.** *The elements in \mathcal{R} are total valuation rings if \mathcal{R} contains one total valuation ring. In this case Q is a division ring.* **Theorem.** $|\mathcal{R}| = 1$ if \mathcal{R} contains an invariant total valuation ring.

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Any valuation v (in the sense of Krull [17]) on a commutative field K has an extension to an arbitrary extension field F . If V is the valuation ring of K corresponding to v this means that there exists a valuation ring B of F with $B \cap K = V$. If, in addition, F is a Galois extension of K then $\{\sigma(B) | \sigma \in G(F/K)\}$ is exactly the set of valuation rings of F that intersect with K in V , where $G(F/K)$ is the Galois group of F over K [11].

In order to consider an analogous noncommutative situation one must decide on what it means to have a valuation on a division ring. Schilling [25] extended Krull's definition as follows: A valuation v on a division ring D is a mapping v from $D^* = D \setminus \{0\}$ onto an ordered group such that $v(ab) = v(a) + v(b)$ and $v(a + b) \geq \min\{v(a), v(b)\}$ for a, b in D^* (see also [26]).

Such valuations on D correspond to subrings B of D with the following two properties:

- (T) If x is in D , x not in B then x^{-1} is in B ; we say B is a *total valuation ring* of D .
- (I) $dBd^{-1} = B$ for all $d \neq 0$ in D , i.e. B is *invariant*.

Received by the editors June 14, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 16A40, 16A39, 16A10, 16A14, 16A16; Secondary 12E15, 12J20, 13A18, 13B02, 13B05.

Key words and phrases. Valuation ring, simple artinian algebra, division ring, extensions, completions, Galois theory, localization, Bezout order.

The first author is supported in part by NSERC.

One can then consider the problem of describing the set of all invariant total valuation rings B in the division ring D with $B \cap K = V$ where V is a given valuation ring in the center K of D and $[D : K] < \infty$. The example of the quaternions H over the rationals shows that such an extension (even if it is not invariant) does not exist in general (only the 2-adic valuation can be extended) and Wadsworth in [28], generalizing a result by Cohn [8], showed that at most one such B exists.

It was proved in [5] that total valuation rings in D that extend the same V are conjugate and that their number is $\leq \sqrt{[D : K]}$. Total valuation rings were also studied by Mathiak [20, 21].

Commutative valuation rings can be described as the elements with finite values under a place mapping. Dubrovin in [9], replacing the fields by simple artinian algebras, gives the following definition: A *subring* R of a simple artinian algebra Q is a Q -valuation ring if R contains an ideal M such that R/M is simple artinian and for each element q in $Q \setminus R$ there exist elements r_1, r_2 in R with $r_1 q, q r_2$ in $R \setminus M$.

A D -valuation ring R , where D is a division ring, is a total valuation ring of D if and only if R/M is a division ring (Lemma 2.2). In §2 of the present paper we will recall some facts about Q -valuation rings and prove in §3 the following

Theorem (Dubrovin). *For every valuation ring V in the center K of a simple algebra Q , finite dimensional over K , there exists a Q -valuation ring R with $R \cap K = V$.*

The proof given here will avoid the transfinite argument used in [10].

The final two sections deal with the set \mathcal{R} of Q -valuation rings that extend a valuation ring in the center K of a finite dimensional simple algebra Q . It is proved in §5 that the elements in \mathcal{R} are conjugate in Q if V has finite rank. If V has rank 1 a real norm function is introduced in Q , that extends the given valuation on V and leads to a completion of Q . In §4 it is shown that $|\mathcal{R}| = 1$ (and Q is a division ring) if \mathcal{R} contains an invariant total valuation ring. All elements in \mathcal{R} are total if one is total.

Dubrovin's extension theorem does not fully generalize the commutative theorem cited at the beginning. Even the condition that K is in the center of Q is necessary: No valuation ring in $\mathbb{Q}(i)$ can be extended by an H -valuation ring in H except the extension of the 2-adic valuation.

Finally, we mention two related results. Cohn and Mahdavi-Hezavehi in [7] consider invariant total valuations with abelian value groups and give conditions for such extensions to be possible. In [4] conditions on the chain ring R_0 , with monomorphism σ , and σ -derivation δ are given that ensure that a total valuation ring R exists in the Ore extension $D(x, \sigma, \delta)$, with $R_0 = R \cap D$, where D is the skewfield of quotients of R_0 .

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Rings that can be considered as the noncommutative equivalent of commutative valuation rings occur for example as coordinate rings of Hjelmslev planes [16], as localizations of rings with a distributive lattice of right ideals [3, 13], in the construction of division rings [1, 6, 15, 19, 22, 27] or as components in structure theorems for certain classes of rings [12, 23].

Rings R in the above list are all right chain rings with possibly zero divisors, i.e. for a, b in R either $aR \subseteq bR$ or $bR \subset aR$ holds. A ring that is a right as well as a left chain ring is called a chain ring. The total valuation rings defined in section 1 are exactly the chain domains. A ring is called right invariant if all its right ideals are two sided and invariant if it is right and left invariant.

Even though extensions of valuations are of central importance in the commutative case, no general extension theory exists for noncommutative valuations. The problem can be (somewhat loosely) formulated as follows: Let \mathcal{V} be a class of rings that serve as noncommutative valuation rings. For example, the class of right chain rings, of total valuation rings, of invariant total valuation rings, of abelian or real invariant total valuation rings. We assume that for every R in \mathcal{V} there exists a certain ring $Q(R)$ of fractions such that $Q(R)$ is in a class \mathcal{F} consisting of the rings T with $Q(T) = T$.

A ring R in \mathcal{V} is called an extension in T of R_0 in \mathcal{V} if T is in \mathcal{F} with $T = Q(R) \supseteq Q(R_0)$ and $R \cap Q(R_0) = R_0$.

Problem. For a given R_0 and T describe the set of all extensions of R_0 in T .

Generalizing the notion of right chain rings to that of right n -chain rings R in an overring Q —i.e. for any set of $n + 1$ elements a_0, \dots, a_n in Q there exists a j with $a_j R \subseteq \sum a_i R$, $i \neq j$ —Dubrovin [9] arrives at the following definition of a Q -valuation ring, which defines a class of rings which is closed under Morita equivalence and rich enough to allow a very interesting extension theory.

Definition 2.1. A subring R of a simple artinian algebra Q is called a Q -valuation ring if R contains an ideal M such that R/M is simple artinian and for any element q in $Q \setminus R$ there exist r_1, r_2 in R with $r_1 q, q r_2$ in $R \setminus M$.

Matrix rings over chain domains, Azumaya algebras over commutative valuation rings [10] and localizations of noncommutative prime Dedekind rings [18] provide examples for Q -valuation rings. A particular example is the ring $R = \mathbb{Z}_3 + \mathbb{Z}_3 i + \mathbb{Z}_3 j + \mathbb{Z}_3 k$ in H , the quaternions over the rationals, which extends the 3-adic valuation ring \mathbb{Z}_3 in \mathbb{Q} .

Lemma 2.2. Let R be a Q -valuation ring with Q and R/M division rings. Then R is a total valuation ring.

Proof. Assume that x is in $Q \setminus R$. Then an element r in R exists with xr in $R \setminus M$, and $xrr' = 1 + m$ for some r' in R, m in M follows. The element $(1 + m)^{-1}$ is in R since otherwise $((1 + m)^{-1} - 1)s = u$ is in $R \setminus M$ for some

s in R . Then, $(1+m)((1+m)^{-1}-1)s = -ms$ is in M ; but this element is also equal to $(1+m)u = u + mu$ in $R \setminus M$, a contradiction. It follows that x^{-1} is in R and R is total.

We will now collect some results from [9 and 10] that will be used in the following sections.

Let R be a Q -valuation ring as in Definition 2.1. It follows from [9] that then $M = J(R)$ (the Jacobson radical of R), that every finitely generated right or left ideal in R is principal (i.e. R is a Bezout ring) and that R is an order in Q .

The converse also holds:

Theorem 2.3 [9]. *The subring R of the simple artinian algebra Q is a Q -valuation ring if and only if $R/J(R)$ is simple artinian and R is a Bezout order of Q .*

It follows that a right chain ring R which is a Q -valuation ring is also a domain, since otherwise the set I of elements in R with nonzero left annihilator is an ideal of R consisting of zero divisors only. In that case Q is a division ring and R is a chain domain, i.e. a total valuation ring, since R is a right and left order in a division ring.

Every simple artinian algebra Q is isomorphic to a matrix ring $(D)_n$ for a division ring D and some n (see [24]). The next result describes the $(D)_n$ -valuation rings as being isomorphic to matrix rings over D -valuation rings. Our proof is somewhat different from the one given in [9].

Theorem 2.4. *Let R be a $(D)_n$ -valuation ring with D a division ring. Then there exists a D -valuation ring T in D and a regular element d in $(D)_n$ with $d^{-1}Rd = (T)_n$.*

Proof. Let e_{ij} be the matrix unit in $Q = (D)_n$ with 1 in the $(i-j)$ -position and zero elsewhere. Consider $\sum_{i,j} Re_{ij} = L$ and L is a cyclic left R -module in Q , using Theorem 2.3, and $L = Rd$ for some d in Q follows. The element d is regular in Q since 1 is in L .

We show that $d^{-1}Rd = d^{-1}L = \sum_{i,j} d^{-1}Re_{ij}$ contains all the matrix units e_{kt} .

The identity 1 is contained in $d^{-1}Rd$, hence $1 = \sum_{i,j} d^{-1}r_{ij}e_{ij}$ for some r_{ij} in R and $e_{kt} = 1e_{kt} = \sum_i d^{-1}r_{ik}e_{it} \in d^{-1}Rd$. This implies $d^{-1}Rd = (T)_n$ for the subring $T = \{a_{11} \in D | (a_{ij}) \in d^{-1}Rd\}$ of D .

It remains to prove that T is a D -valuation ring. Since $(T)_n/J((T)_n) \cong (T/J(T))_n$ and $R/J(R)$ is simple artinian it follows that $T/J(T)$ is simple artinian. To check the second condition let $q \in D \setminus T$ and $qe_{11} \in Q \setminus (T)_n$. Then $d(qe_{11})d^{-1} \in Q \setminus R$ and $dqe_{11}d^{-1}r \in R \setminus J(R)$ for some r in R . It follows that $qe_{11}(d^{-1}rd) \in (T)_n \setminus J((T)_n)$ and $qs_{1i} \in T \setminus J(T)$ for at least one entry $s_{1i} \in T$ in the first row of the matrix $d^{-1}rd \in (T)_n$.

We later need the following results [9, Lemma 7].

Lemma 2.5. *Let R be a Q -valuation ring, Q simple artinian. Then $O_r(J(R)) = O_e(J(R)) = R$.*

Here, $O_r(I) = \{q \in Q \mid Iq \subseteq I\}$ is the right order of an additive subgroup I of Q . The left order $O_e(I)$ is defined similarly.

For the rest of this section let Q be a simple algebra, finite dimensional over its center K . We recall the following results:

Proposition 2.6 [9]. *Let R be a Q -valuation ring and $sR \supseteq Rs$ or $Rs \supseteq sR$ for some s in Q . Then $sR' = R's$ for any ring R' with $R \subseteq R' \subseteq Q$. Further, $s^n R = kR$ for some natural number n and some k in K .*

The next result shows that the assumption of the previous proposition holds for finitely generated R -ideals if R is a maximal Q -valuation ring.

Proposition 2.7 [9]. *Let R be a Q -valuation ring and T a finitely generated R -ideal. Then $T = sS = Ss$ for some s in Q and $S = O_r(T) = O_e(T)$.*

Here, an additive subgroup I of Q is called a right R -ideal if $IR = I$, I contains a regular element and $aI \subseteq R$ for some regular element a in Q . An R -ideal is a right and left ideal.

The next result [10, Theorem 1] describes the relationship between prime ideals in R (not necessarily completely prime) and prime ideals in $V = R \cap K$. As in the commutative case, the overrings of R can be described as localizations of R on its prime ideals.

Theorem 2.8. *Let R be a Q -valuation ring for a simple algebra Q , finite dimensional over its center K . Then the following statements are true:*

- (i) $R \cap K = V$ is a valuation ring of K .
- (ii) The mapping $\varphi(P) = P \cap K$ is an injective mapping from the set $\text{spec}(R)$ of prime ideals of R onto $\text{spec}(V)$.
- (iii) The mapping $\psi(S) = S \cap K$ from the set of rings S with $R \subseteq S \subseteq Q$ is an injective mapping onto the set of rings between V and K .
- (iv) The set $C(P)$ of elements in R which are regular mod P is an Ore set in R for P in $\text{spec}(R)$. The localization R_P exists and $R_P = RV_{\varphi(P)}$.
- (v) The mapping $\chi(P) = R_P$ is an injective mapping from $\text{spec}(R)$ onto the set of rings between R and Q .

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Let Q be a central simple algebra over the center K with $[Q : K] = n$ and V a valuation ring in K . We want to show that there exists a Q -valuation ring R with $R \cap K = V$. This result was proved by Dubrovin in [10]; our proof is influenced by this proof, but differs in some essential points.

Lemma 3.1. *Let R be a valuation ring in the simple artinian ring Q and q an element in Q not in R . Then there exists j in $J(R)$ with qj in $R \setminus J(R)$.*

To prove this assume $qJ(R) \cap R \subseteq J(R)$. If qm is not in R for some m in $J(R)$ then $qmr \in R \setminus J(R)$ for some r in R —a contradiction. We have $qJ(R) \subseteq J(R)$ and $q \in O_e(J(R))$ —a contradiction to Lemma 2.5.

This result is used to show that certain subrings of a Q -valuation ring R are again Q -valuation rings, which in turn will be used in an induction argument to prove the main result.

Lemma 3.2. *Let R be a Q -valuation ring in the simple artinian ring Q and S a subring of R with $S \supseteq J(R)$. If $\bar{S} = S/J(R)$ is an $R/J(R)$ -valuation ring then S is a Q -valuation ring.*

Proof. Let φ be the canonical homomorphism from S to $S/J(R)$ and denote $\varphi^{-1}(J(S/J(R)))$ by M .

We have $S/M \cong \bar{S}/J(\bar{S})$ which is simple artinian since \bar{S} is a valuation ring.

Next, let $q \in Q \setminus S$. If $q \in R \setminus S$ then there exists s in S with $\varphi(q)\varphi(s) = \varphi(r)$ where r is in $S \setminus M$ and φ is extended to R to denote the mapping modulo $J(R)$. The element $qs - r$ is contained in $J(R)$ and $qs \in S \setminus M$ follows. If $q \in Q \setminus R$, then by Lemma 3.1 there exists j in $J(R)$ with $q_0 = qj \in R \setminus J(R)$. We can assume that $\bar{S} = S/J(R) \neq R/J(R) = \bar{R} = (D_0)_m$ for a division ring D_0 . By Theorem 2.4 there exists a regular element d in \bar{R} with $d^{-1}\bar{S}d = (T_0)_m$ for a D_0 -valuation ring T_0 , and $d^{-1}\bar{M}d = (J(T_0))_m$ follows. Since $d^{-1}\bar{q}_0d \neq 0$ in $\bar{R} = (D_0)_m$, there exists an element r_1 in R with $d^{-1}\bar{q}_0dd^{-1}\bar{r}_1d = d^{-1}\bar{q}_0\bar{r}_1d \notin (J(T_0))_m$ and hence $\bar{q}_0\bar{r}_1 \in \bar{R} \setminus \bar{M}$. By the above argument there exists an element s in S with $qjr_1s \in S \setminus M$ and $jr_1s \in S$; it follows that S is a Q -valuation ring.

We need one more technical lemma.

Lemma 3.3. *Let Q be a finite dimensional simple algebra over its center K and V a valuation ring of K . Assume that R is a Q -valuation ring with $R \cap K = V$. Then there exists for any q in Q a nonzero element v in V with qv in R .*

Proof. We assume first that Q is a division ring and $q \notin R$. Then there exists r ($\neq 0$) in R with $qr \in R \setminus J(R)$. Let $f(x) \in V[x]$ be a polynomial of minimal degree in $V[x]$ with $f(r) = 0$, $f(x) = a_n x^n + \cdots + a_1 x + a_0$, and $a_0 \neq 0$. We have $qa_0 = -qr(a_n r^{n-1} + \cdots + a_1) \in R$.

If now $Q = (D)_n$, then we can assume $R = (T)_n$, T a D -valuation ring, $q = (q_{ij})$, $q_{ij} \in D$ (Theorem 2.4). Applying the first part of the proof there exists v in V , $v \neq 0$, with $q_{ij}v \in T$ for all i, j and $qv \in R$ follows.

The next result shows that rank 1 valuation rings in K can be extended.

Lemma 3.4. *Let Q be a simple algebra finite dimensional over its center K , V a maximal (i.e. rank 1) valuation ring of K . Then there exists a Q -valuation ring R with $R \cap K = V$.*

Proof. Let \tilde{K} be the completion of K with respect to the valuation defined by V (i.e. with respect to the topology defined by the ideals of V) and denote with \tilde{V} the corresponding completion of V . Then $C = Q \otimes_K \tilde{K}$ is a finite dimensional central simple algebra over \tilde{K} [24] and we identify V, K, \tilde{K} and Q with their canonical images in $C = Q\tilde{K}$. There is an isomorphism φ from C onto $(D)_n$ for some division ring D and $\varphi(\tilde{K})$ is a subfield of D .

By a well-known result [26, p. 53] there exists a total valuation subring T of D with $T \cap \varphi(\tilde{K}) = \varphi(\tilde{V})$ and $S' = (T)_n$ is a $(D)_n$ -valuation ring with $S' \cap \varphi(\tilde{K}) = \varphi(\tilde{V})$. We denote $\varphi^{-1}(S')$ by S and S is a C -valuation ring with $S \cap \tilde{K} = \tilde{V}$.

We denote $S \cap Q$ by R and $J(S) \cap Q$ by M and want to show that R is a Q -valuation ring with $R \cap Q = V$. We claim first that $R + J(S) = S$.

The left-hand side is clearly contained in S . Let $s \in S$ and

$$s = q_1 k_1 + \cdots + q_n k_n, \quad k_i \in \tilde{K}, \quad q_i \in Q.$$

By Lemma 3.3 there exists $0 \neq a$ in \tilde{V} with $q_i a \in J(S)$. Since K is dense in \tilde{K} , there exist elements k'_i in K with $\tilde{v}(k_i - k'_i) \leq \tilde{v}(a)$ where \tilde{v} is the valuation corresponding to \tilde{V} . We obtain

$$s = \sum q_i k'_i + \sum q_i (k_i - k'_i) \quad \text{where} \quad \sum q_i (k_i - k'_i) \text{ is in } J(S),$$

$\sum q_i k'_i$ is in Q and also in S and hence in R , which proves the claim.

We apply this and obtain

$$\frac{R}{M} = \frac{Q \cap S}{Q \cap J(S)} \cong \frac{(Q \cap S) + J(S)}{J(S)} = \frac{S}{J(S)}$$

is simple artinian.

Let q be in $Q \setminus R$, then $q \in C \setminus S$ and s exists in S with $qs \in S \setminus J(S)$. As before: $s = \sum_1^n q_i k_i$, $q_i \in Q$, $k_i \in \tilde{K}$. By Lemma 3.3 there exists an element $0 \neq a$ in \tilde{V} with $qq_i a$, $q_i a \in J(S)$ for all i . Further, there exist elements k'_i in K with $\tilde{v}(k'_i - k_i) \leq \tilde{v}(a)$.

We have $s = \sum q_i k'_i + \sum q_i (k_i - k'_i)$ where the first sum is an element in $Q \cap S$ and the second an element in $J(S)$. Hence, $q \sum q_i k'_i = qs - \sum qq_i (k_i - k'_i)$ is in $(Q \cap S) \setminus (Q \cap J(S))$ since the last sum is in $J(S)$ and qs is in $S \setminus J(S)$. We conclude that $R = Q \cap S$ is a Q -valuation ring with $R \cap K = V$.

This last lemma can be applied immediately to prove the next result in case the valuation ring V in K has a minimal prime ideal P .

Lemma 3.5. *Let Q be a finite dimensional central simple algebra over K, V ($\neq K$) a valuation ring of K . Then there exists a Q -valuation ring R with $R \neq Q$ and $R \cap Q \supseteq V$.*

Proof. If V contains a minimal prime ideal P one localizes V at P , applies Lemma 3.4 and obtains the result. We now assume that V does not contain a minimal prime ideal.

Let $\{m_1, m_2, \dots, m_n\}$ be any K -basis of Q with $m_1 = 1$. We have $m_i m_j = \sum a_{ijt} m_t$, $a_{ijt} \in K$, and an element $0 \neq c$ exists in K with $ca_{ijt} \in V$ for all i, j, t . Consider the K -basis $\{1, a_2, \dots, a_n\}$ with $a_i = cm_i$, $i = 2, \dots, n$.

We now use the following result of Azumaya [2, Theorem 12]: A K -algebra A with basis $\{a_1, \dots, a_n\}$ over K is centrally simple over K if and only if the $n \times n$ matrix $N = (c_{ij})$, $c_{ij} = a_j a_i$ has an inverse in $(A)_n$.

If we apply this result to Q and the basis $\{1, a_2, \dots, a_n\}$ as above we obtain an inverse $N^{-1} = (b_{ij})$ of N in $(Q)_n$. We have $b_{ij} = \sum_t \beta_{ijt} a_t$ with β_{ijt} in K , $a_1 = 1$. Since V has no minimal prime ideal there exists a localization $V' \neq K$ of V with $\beta_{ijt} \in V'$ for all i, j, t . We denote with M' the maximal ideal $J(V')$ of V' . Then $R = \sum_{i=1}^n V' a_i$ is a subring of Q and $M = \sum M' a_i$ is an ideal of R . It follows that $\{\bar{a}_1, \dots, \bar{a}_n\}$ is a V'/M' -basis for R/M and the matrix $\bar{N} = (\bar{c}_{ij})$ has an inverse in $(R/M)_n$ —where the bar denotes the mapping modulo M . By the above-mentioned result of Azumaya it follows that R/M is simple artinian. Finally, for each q in $Q \setminus R$ there exists an s in V' with $qs \in R \setminus M$ — R is a Q -valuation ring with $R \cap Q \supseteq V$ and $R \neq Q$.

Lemma 3.6. *Let D be a division ring with center K and $[D : K] = n$. Let V be a valuation ring in K , B_1 and B_2 two distinct total valuation rings in D with $B_1 \cap K = V = B_2 \cap K$. Then*

$$\left[\frac{B_1}{M_1} : \frac{V}{M} \right] + \left[\frac{B_2}{M_2} : \frac{V}{M} \right] \leq n \quad \text{where } J(B_i) = M_i, J(V) = M.$$

To prove this, observe that $B_i \not\subseteq B_j$ using Theorem 2.8. We will give a direct proof without applying this result. If $B_1 \supset B_2$ then $B_1 \supset B_2 \supset M_2 \supset M_1$ and M_2, M_1 are completely prime ideals in B_2 .

Let $m_2 \in M_2 \setminus M_1$ and elements v_i exist in V , not all in M , with

$$v_k m_2^k + \dots + v_1 m_2 + v_0 = 0, \quad v_0 \neq 0.$$

It follows that $v_0 = -m_2(v_1 + v_2 m_2 + \dots + v_k m_2^{k-1}) \in M_2 \cap K = M$, hence $v_0 \in M_1$ and $v_1 + v_2 m_2 + \dots + v_k m_2^{k-1} \in M_1 \subset M_2$. As before, $v_1 \in M_2 \cap K = M \subseteq M_1$. Iterating this process leads to the contradiction $v_i \in M_2$ for all i .

This shows that $B_i \not\subseteq B_j$. Therefore, there exists $x'_1 \in B_1, x'_1 \notin B_2$. We can assume that x'_1 is not in M_1 , otherwise replace x'_1 by $1 + x'_1$. The element $x_1 = x'^{-1}_1$ is then contained in $B_1 \setminus M_1$ and in M_2 . Similarly, there exists an element $x_2 \in (B_2 \setminus M_2) \cap M_1$.

Let a_1, \dots, a_r be in B_1 such that $\{a_i + M_1 | i = 1, \dots, r\}$ is a basis of B_1/M_1 over V/M . We can assume that $a_i B_2 \supseteq a_i M_2$, $i = 1, \dots, r$ and after replacing a_i by $x_1 a_i^{-1} a_i$ we can assume that the a_i are all in M_2 .

Similarly, we can find a basis $\{b_j + M_2 | j = 1, \dots, s\}$ for B_2/M_2 over V/M with $b_j \in B_2 \cap M_1$.

If the set $\{a_1, \dots, a_r, b_1, \dots, b_s\}$ is linearly dependent then there exists an equation $\sum w_i a_i + \sum w'_j b_j = 0$, $w_i, w'_j \in V$, not all in M . This leads to a contradiction.

Lemma 3.7. *Let Q be a central simple algebra over K with $[Q : K] = n$ and R a Q -valuation ring with $R \cap K = V$, V a valuation ring of K . Then $[R/J(R) : V/J(V)] \leq n$.*

It is only necessary to observe that $\{r_i | i = 1, \dots, t\}$, r_i in R , is linearly independent over K if $\{r_i + J(R) | i = 1, \dots, t\}$ is a basis for $R/J(R)$ over $V/J(V)$.

We can now prove the main result of this section.

Theorem 3.8. *Let Q be a simple algebra with center K and $[Q : K] = n$ and V a valuation ring in K . Then there exists a Q -valuation ring R with $R \cap K = V$.*

Proof by induction on n , where the case $n = 1$ is trivial. In the general case let \mathcal{R}' be the set of all Q -valuation rings R in Q with $Q \neq R$ and $R \cap K \supseteq V$. (We can assume that $V \neq K$.)

The set \mathcal{R}' is not empty by Lemma 3.5. We assume first that there exists a ring R in \mathcal{R}' with $R/J(R) \cong (D_0)_m$ and D_0 a division ring, $m > 1$. Then we can assume $R/J(R) = (D_0)_m$ and obtain, using Lemma 3.7,

$$\left[D_0 : \frac{R \cap K}{J(R) \cap K} \right] < \left[\frac{R}{J(R)} : \frac{R \cap K}{J(R) \cap K} \right] \leq n.$$

We have $R \cap K \supseteq V \supset J(V) \supseteq J(R) \cap K = J(R \cap K)$. By induction there exists a D_0 -valuation ring T_0 with

$$T_0 \cap \left(\frac{R \cap K}{J(R) \cap K} \right) = \frac{V}{J(R) \cap K} \quad \text{and} \quad (T_0)_m = \frac{S}{J(R)}$$

is an extension of $V/(J(R) \cap K)$ in $R/J(R)$ for a subring S of R with $S \supseteq J(R)$. By Lemma 3.2 it follows that S is a Q -valuation ring and $S \cap K = V$.

We must now consider the case where $J(R)$ is completely prime in R for every R in \mathcal{R}' . It follows (Lemma 2.2, Theorem 2.4) that $Q = D$ is a division ring and R is a total valuation ring in D for every R in \mathcal{R}' .

If there exists an R in \mathcal{R}' with

$$\left[\frac{R}{J(R)} : \frac{R \cap K}{J(R) \cap K} \right] < n$$

we can use induction to extend $V/(J(R) \cap K)$ to $R/J(R)$ and obtain with Lemma 3.2 a D -valuation ring S with $S \cap K = V$.

We are left with the case that all R in \mathcal{R}' are total valuation rings and

$$\left[\frac{R}{J(R)} : \frac{R \cap K}{J(R) \cap K} \right] = n \quad \text{for all } R \text{ in } \mathcal{R}'.$$

We will prove below that then $R_1 \subseteq R_2$ or $R_2 \subseteq R_1$ for any $R_1, R_2 \in \mathcal{R}'$.

Assuming this fact let R_0 be the intersection of all R_i in \mathcal{R}' and R_0 is the minimal element in \mathcal{R}' . If $R_0 \cap K \supset V$, then

$$\frac{R_0}{J(R_0)} \supset \frac{R_0 \cap K}{J(R_0) \cap K} \supset \frac{V}{J(R_0) \cap K}$$

and by Lemma 3.5 an $R_0/J(R_0)$ valuation ring $S/J(R_0) \neq R/J(R_0)$ exists with

$$\frac{S}{J(R_0)} \cap \frac{R_0 \cap K}{J(R_0) \cap K} \supseteq \frac{V}{J(R_0) \cap K}$$

and $R_0 \supset S$ and $S \cap K \supseteq V$ is a contradiction.

To prove the above claim let $R_1, R_2 \in \mathcal{R}'$ and we can assume that $R_1 \cap K \subseteq R_2 \cap K$ (these rings are overrings of V). We will show that $R_1 \subseteq R_2$.

The ring $(R_2 \cap K)R_1$ is an overring of R_1 . We show that $(R_2 \cap K)R_1 \cap K = R_2 \cap K$. Let $s_1 r_1 + s_2 r_2 + \dots + s_t r_t = k \in K$ with $s_i \in R_2 \cap K$, $r_i \in R_1$ for all i . If $k \notin R_2 \cap K$ then k^{-1} and hence $k^{-1} s_i \in J(R_2 \cap K) \subseteq J(R_1 \cap K)$, a contradiction, since then $\sum k^{-1} s_i r_i = 1 \in J(R)$. We can apply Lemma 3.6 to $B_1 = (R_2 \cap K)R_1$ and $B_2 = R_2$ and obtain $B_1 = B_2$ and $R_1 \subseteq R_2$.

4

In this section let D be a division ring finite dimensional over its center K and V a valuation ring in K . Let \mathcal{R} be the set of D -valuation rings R with $R \cap K = V$ and we know by Theorem 3.8 that $|\mathcal{R}| \geq 1$. We show that $|\mathcal{R}| = 1$ if \mathcal{R} contains an invariant total valuation ring and that every element in \mathcal{R} is total if one of them is.

Lemma 4.1. *Let D be a division ring finite dimensional over its center K . Let V be a valuation ring of K . Assume that the set \mathcal{R} of extensions of V in D contains an invariant total valuation ring B , then $\mathcal{R} = \{B\}$.*

For a proof assume that R is a D -valuation ring with $R \cap K = V$. We want to show that $R = B$ and claim first that $J(R) \subseteq J(B)$. To prove this let $0 \neq m$ be an element in $J(R)$ and let $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ be the minimal polynomial of m over K . Assume that m^{-1} is in B hence integral over V [5, Theorem 3]. This implies that $x^n + a_0^{-1}a_1x^{n-1} + \dots + a_0^{-1}a_{n-1}x + a_0^{-1} = g(x)$ is the minimal polynomial of m^{-1} over K and $a_0^{-1}a_i \in V$ follows for $i = 0, \dots, n$ with $a_n = 1$. Using $f(m) = 0$ we obtain

$$a_0^{-1}m^n + a_0^{-1}a_{n-1}m^{n-1} + \dots + a_0^{-1}a_1m = -1 \in J(R), \quad \text{a contradiction.}$$

Hence, $m^{-1} \notin B$ which implies $m \in J(B)$ and $J(R) \subseteq J(B)$ follows. Next, assume that $B \neq R$. Then B and R are two distinct D -valuation rings which are extensions of V and by Theorem 2.8 it follows that $R \not\subseteq B$. Let $r \in R$, $r \notin B$. If $[D : K] = n$ then $1, r, r^2, \dots, r^n$ are linearly dependent over K and elements k_i exist in K , not all zero, with $\sum_0^n k_i r^i = 0$. Hence, there exist

$j > i$ with $k_i r^i \neq k_j r^j$ and $k_i r^i B = k_j r^j B$ and $r^{j-i} B = k_j^{-1} k_i B$ follows, i.e. $r^i B = k B$ for $t = j - i$ and an element k in K .

Since $r \notin B$, we have $r^i \notin B$, $k \notin B$ and $k^{-1} \in J(B) \cap K = J(R) \cap K = J(V)$. The element $k^{-1} r^i$ is a unit in B and $k^{-1} r^i \notin J(B)$. But $k^{-1} r^i \in J(R) \subseteq J(B)$, a contradiction that proves $R = B$.

Corollary 4.2. *Let D be a finite dimensional division algebra over its center K . Assume that K is complete with respect to a rank 1 valuation with corresponding valuation ring V . Then there exists exactly one D -valuation ring R with $R \cap K = V$.*

Proof. It follows from [26, p. 53] that there exists an invariant total valuation ring B of D with $B \cap K = V$ and the statement follows from Lemma 4.1.

The next result deals with the case that the set \mathcal{R} of extensions contains a total valuation ring.

Theorem 4.3. *Let D be a division ring finite dimensional over its center K and V a valuation ring in K . Assume that the set \mathcal{R} of extensions of V in D contains a total valuation ring B . Then every extension R in \mathcal{R} is a total valuation ring.*

Proof. We use induction on $n = [D : K]$ and the case $n = 1$ is trivial. In the general case assume first that B is the only total valuation ring extending V . Then B is also invariant and the result follows from Lemma 4.1.

If we assume that B is not the only total valuation ring in \mathcal{R} , there exists [5, Lemma 4] an invariant total valuation ring T in D , $T \neq D$, that contains all total valuation rings in \mathcal{R} . Let R be an arbitrary element in \mathcal{R} . It follows from Theorem 2.8 that there exists in R a prime ideal P with $P \cap V = J(T) \cap V$ and a D -valuation ring $R_1 \supseteq R$ with $J(R_1) = P$ and $R_1 \cap K = T \cap K$. Since T is invariant, Lemma 4.1 implies $T = R_1$ and $P = J(T)$. The division ring $T/J(T)$ contains $R/J(T)$ and $B/J(T)$ as $T/J(T)$ -valuation rings which are extensions of $V/(J(T) \cap K)$.

By Lemma 5 in [5] and our assumption it follows that $(T \cap K)/(J(T) \cap K)$ is properly contained in the center Z of $T/J(T)$ and $[T/J(T) : Z] < n$.

The intersection $R/J(T) \cap Z$ is a valuation ring of the commutative field Z . Again by Lemma 5 in [5] it follows that the separable closure S of $(T \cap K)/(J(T) \cap K)$ in Z is a Galois extension such that the $(T \cap K)/(J(T) \cap K)$ -automorphisms of S are induced by inner automorphisms of D .

Hence, there exists an element $0 \neq d$ in D with $(dBd^{-1}/J(T)) \cap Z = (R/J(T)) \cap Z$ and $dBd^{-1}/J(T)$ is an extension of $R/J(T) \cap Z$ which is a total valuation ring in $T/J(T)$. By induction $R/J(T)$ is a total valuation ring in $T/J(T)$, the ring T is total and hence R is total which proves the theorem.

5

It is shown in this section that the elements in the set \mathcal{R} of Q -valuation rings R with $R \cap K = V$ are conjugate in Q if Q is a simple algebra finite dimensional over its center K and V is a valuation ring of K of finite rank.

The rank 1 case is proved by using the completion of Q with respect to the topology given by the finitely generated ideals in a Q -valuation ring R . A norm can be defined on Q that extends the valuation of K given by V .

Let Q be a central simple algebra finite dimensional over its center K . Let R be a maximal Q -valuation ring and it follows from Theorem 2.8 that this is the case if and only if (0) and $J(R)$ are the only prime ideals in R if and only if $R \cap K = V$ is a maximal (i.e. rank 1) valuation ring in K . Denote with v the corresponding valuation of K whose value group is isomorphic to a subgroup of (\mathbb{R}^+, \circ) the ordered group of positive real numbers under multiplication.

If I is a finitely generated R -ideal in Q , then $O_e(I) = R = O_r(I)$ and $I = rR = Rr$ for some regular element in Q (Propositions 2.6 and 2.7). It follows that the set $G = \{rR \mid r \text{ regular in } Q \text{ and } rR = Rr\}$ is an ordered group.

We show that G can be considered as a subgroup of (\mathbb{R}^+, \circ) by checking the archimedean property for G : Let $sR, s'R \in G$, $sR, s'R \subseteq J(R)$. By Proposition 2.6 there exist natural numbers n and n' and k, k' in $J(V)$ with $s^n R = kR$, $s'^{n'} R = k'R$. But V is maximal and there exists m with $k^m \in k'V \subseteq k'R$ and $s^{nm} R = k^m R \subseteq k'R = s'^{n'} R \subseteq s'R$ follows.

The mapping N from Q to $\hat{G} = G \cup \{0\}$ defined by $N(q) = RqR$ considered as element of \hat{G} has the following properties:

- (i) $N(q) = 0$ if and only if $q = 0$.
- (ii) $N(kq) = v(k)N(q)$ for $0 \neq k$ in K , q in Q .
- (iii) $N(q + q') \leq \max\{N(q), N(q')\}$.

The mapping N defines a norm on Q , which is a topological vectorspace with respect to N and v . In addition, the following property holds:

- (iv) $N(qq') \leq N(q)N(q')$ for q and q' in Q and Q is a topological ring.

Let \tilde{Q} be the completion of Q with respect to N (i.e. $\tilde{Q} = \hat{Q}/A$ where \hat{Q} is the ring of all Cauchy sequences and A is the two-sided ideal of all null sequences in \hat{Q}). We also say that \tilde{Q} is the completion of Q with respect to R . Let \tilde{K} be the completion of K with respect to v and we consider \tilde{K} as a subring of the center of \tilde{Q} .

Let $0 \neq q \in \tilde{Q}$, $q = (a_i) + A$, $(a_i) \notin A$. Then there exists a natural number n_0 with

$$N(a_n) = N(a_0) > 0 \quad \text{for all } n \geq n_0.$$

We define $\tilde{N}(q) = N(a_0)$, $\tilde{N}(0) = 0$ and \tilde{N} defines a norm on \tilde{Q} considered as vectorspace over \tilde{K} , in fact \tilde{N} satisfies (i)–(iv) for \tilde{Q} and \tilde{K} . The restriction of \tilde{N} to \tilde{K} is a valuation \tilde{v} that extends v and \tilde{K} is complete with respect to \tilde{v} . Q and \tilde{K} can be considered as subrings of \tilde{Q} and the mapping

$\varphi(\sum(g_i \otimes k_i)) = \sum q_i k_i$ defines an K -algebra homomorphism from $Q \otimes_K \tilde{K}$ to \tilde{Q} and φ is injective since $Q \otimes_K \tilde{K}$ is a simple \tilde{K} -algebra. The image $Q\tilde{K}$ of φ is a finite dimensional vector space over the complete field \tilde{K} and \tilde{N} induces a norm on $Q\tilde{K}$. Every element q in \tilde{Q} can be considered as the limit of a sequence (a_i) , a_i in Q . But (a_i) is also a Cauchy sequence in the complete \tilde{K} -vectorspace $Q\tilde{K}$ and hence has a limit in $Q\tilde{K}$. This implies q in $Q\tilde{K}$ and shows that $Q \otimes_K \tilde{K}$ and \tilde{Q} are isomorphic; \tilde{Q} is a central simple algebra over \tilde{K} . The set $\tilde{R} = \{q \in \tilde{Q} | \tilde{N}(q) \leq \tilde{N}(1)\}$ is a subring of \tilde{Q} and $\tilde{M} = \{q \in \tilde{Q} | \tilde{N}(q) < \tilde{N}(1)\}$ is an ideal of \tilde{R} . Since Q is dense in \tilde{Q} , there exists for every q in \tilde{R} an element r in R with $\tilde{N}(q - r) < \tilde{N}(1)$ and $\tilde{R}/\tilde{M} \cong R/J(R)$ follows, i.e. \tilde{R}/\tilde{M} is simple artinian.

Finally, for each $q \in \tilde{Q} \setminus \tilde{R}$ there exist r, r' in \tilde{R} with $qr', rq \in \tilde{R} \setminus \tilde{M}$ since R is a Q -valuation ring and Q is dense in \tilde{Q} .

We summarize the above results.

Proposition 5.1. *Let Q be a simple algebra, finite dimensional over its center K and R be a maximal Q -valuation ring with $R \cap K = V$. Then V is a rank 1 valuation ring of K . If \tilde{K} is the completion of K with respect to V , then $Q \otimes_K \tilde{K}$ is the completion \tilde{Q} of Q with respect to R and contains a \tilde{Q} -valuation ring \tilde{R} with $\tilde{R} \cap Q = R$ and $\tilde{R} \cap \tilde{K} = \tilde{V}$, the completion of V .*

We show in the next result that the extensions of a rank-1 valuation ring V in the center K of a simple ring Q , $[Q : K] < \infty$, are conjugate in Q .

Theorem 5.2. *Let Q be a simple algebra finite dimensional over its center K and V a rank-1 valuation ring of K . Then any two Q -valuation rings R and R' with $R \cap K = V = R' \cap K$ are conjugate in Q .*

Proof. Let $\tilde{Q} \supseteq \tilde{R}$ and $\tilde{Q}' \supseteq \tilde{R}'$ be the completions of Q with respect to R and R' respectively. By Proposition 5.1 we have $\tilde{Q} \cong Q \otimes_K \tilde{K} \cong \tilde{Q}'$ and can assume the following: $\tilde{Q} = Q\tilde{K}$ contains two \tilde{Q} -valuation rings \tilde{R} and \tilde{R}' with

$$\tilde{R} \cap Q = R, \quad \tilde{R}' \cap Q = R', \quad \tilde{R} \cap \tilde{K} = \tilde{R}' \cap \tilde{K} = \tilde{V}.$$

Further, $Q\tilde{K} \cong (L)_n$ for some division ring L and we can assume $Q\tilde{K} = (L)_n$ with L contained in $Q\tilde{K}$ and finite dimensional over its center \tilde{K} .

By Theorem 2.4 there exist regular elements $u, v \in Q\tilde{K}$ with $u\tilde{R}u^{-1} = (S)_n$, $v\tilde{R}'v^{-1} = (S')_n$ where S, S' are L -valuation rings. Then $S \cap \tilde{K} = u\tilde{R}u^{-1} \cap \tilde{K} = \tilde{R} \cap \tilde{K} = \tilde{R}' \cap \tilde{K} = v\tilde{R}'v^{-1} \cap \tilde{K} = S' \cap \tilde{K}$ and $S = S'$ by Corollary 4.2 which implies $u\tilde{R}u^{-1} = v\tilde{R}'v^{-1}$ and $d\tilde{R}d^{-1} = \tilde{R}'$ for a regular element d in $Q\tilde{K}$.

Let $\{q_1, \dots, q_n\}$ be a K -basis of Q and N the norm on Q induced by R .

$$d = k_1 q_1 + \dots + k_n q_n \quad \text{with } k_i \in \tilde{K}, \quad q_i \in Q.$$

Since K is dense in \tilde{K} there exist elements k'_j in K with

$$\tilde{v}(k_j - k'_j) < N(d^{-1})^{-1} N(q_j)^{-1} \quad \text{for all } j.$$

We have $d = \sum q_i k'_i + \sum q_i (k_i - k'_i)$ and

$$\begin{aligned} N\left(d^{-1}\left(\sum q_i(k_i - k'_i)\right)\right) &\leq N(d^{-1})N\left(\sum q_i(k_i - k'_i)\right) \\ &\leq N(d^{-1})\max\{N(q_j)\tilde{v}(k_j - k'_j)\} \\ &< N(d^{-1})\max\{N(q_j)N(d^{-1})^{-1}N(q_j)^{-1}\} = 1. \end{aligned}$$

We have

$$q = \sum q_i k'_i = d\left(1 - d^{-1}\left(\sum q_i(k_i - k'_i)\right)\right) \in Q$$

and

$$w = 1 - d^{-1}\left(\sum q_i(k_i - k'_i)\right)$$

is a unit in \tilde{R} . Hence, $d\tilde{R}d^{-1} = dw\tilde{R}w^{-1}d^{-1} = q\tilde{R}q^{-1} = \tilde{R}'$ for q in Q . Therefore, $qRq^{-1} = q(\tilde{R} \cap Q)q^{-1} = \tilde{R}' \cap Q = R'$ which proves Theorem 5.2.

Before we can prove that the elements in \mathcal{R} are conjugate in Q in case V has finite rank we need the following result:

Proposition 5.3. *Let Q be a simple algebra finite dimensional over its center K and V a rank 1 valuation ring in K . Let R be a Q -valuation ring with $R \cap K = V$ and let Z be the center of $R/J(R)$ with S the separable closure of $V/J(V)$ in Z .*

Then S is a Galois extension of $V/J(V)$ and every $V/J(V)$ -automorphism of S is induced by an inner automorphism of Q that maps R onto R .

Proof. Let \tilde{K} be the completion of K with respect to (the topology given by the ideals of) V with \tilde{V} the completion of V and $C = Q\tilde{K} \cong Q \otimes_K \tilde{K} \cong (D)_n$ for some division ring D . Let φ be the isomorphism from C to $(D)_n$ and $\varphi(\tilde{K})$ is then the center of $(D)_n$ which can also be considered as the center of D .

Since \tilde{V} is complete of rank 1, $\varphi(\tilde{V})$ has a unique extension B in D (Corollary 4.2) and $(B)_n$ is a $(D)_n$ -valuation ring with $(B)_n \cap \varphi(\tilde{K}) = \varphi(\tilde{V})$.

We have

$$J((B)_n) = (J(B))_n \quad \text{and} \quad (B)_n/J((B)_n) \cong (B/J(B))_n.$$

Hence, if A denotes the center of $B/J(B)$ then A is isomorphic to the center of $(B)_n/J((B)_n)$.

Let S' be the separable closure of $\varphi(\tilde{V})/J(\varphi(\tilde{V}))$ in A . Since B is an invariant total valuation ring of the skew field D , it follows that S' is a Galois extension of $\varphi(\tilde{V})/J(\varphi(\tilde{V}))$ and every $\varphi(\tilde{V})/J(\varphi(\tilde{V}))$ -automorphism of S' is induced by an inner automorphism of D . ((2) in Lemma 5 in [5] remains true without the condition $|\mathcal{B}| > 1$.)

If we denote with S'' the separable closure of $\varphi(\tilde{V})/J(\varphi(\tilde{V}))$ in the center of $(B)_n/J((B)_n)$ —which is isomorphic to A —then S'' is a Galois extension of $\varphi(\tilde{V})/J(\varphi(\tilde{V}))$ and the $\varphi(\tilde{V})/J(\varphi(\tilde{V}))$ -automorphisms of S'' are induced by inner automorphisms of $(D)_n$ that map $(B)_n$ onto $(B)_n$.

The preimage $R_C = \varphi^{-1}((B)_n)$ is a C -valuation ring that extends \tilde{V} . The separable closure S_C of $\tilde{V}/J(\tilde{V})$ in the center of $R_C/J(R_C)$ is a Galois extension of $\tilde{V}/J(\tilde{V})$ and every $\tilde{V}/J(\tilde{V})$ -automorphism of S_C is induced by an inner automorphism of C corresponding to an invertible element d in C with $dR_Cd^{-1} = R_C$.

In the last part of the proof of Theorem 5.2 it was shown that there exist q in Q and j in $J(R_C)$ with $q = d(1 - j)$ and q induces the same automorphism on $R_C/J(R_C)$ as d does.

In the proof of Lemma 3.4 it is shown that $R_C \cap Q = R'$ is a Q -valuation ring and $R' \cap K = V$. From the remarks before Proposition 4.4 it follows that $R_C/J(R_C)$ and $R'/J(R')$ as well as $\tilde{V}/J(\tilde{V})$ and $V/J(V)$ are isomorphic. The statement of Proposition 5.3 is therefore true for R' and hence for R which is conjugate to R' by Theorem 5.2.

Theorem 5.4. *Let Q be a simple algebra finite dimensional over its center K and V a valuation ring in K of finite rank. Then any two Q -valuation rings R and R' with $R \cap K = R' \cap K = V$ are conjugate in Q .*

The theorem will be proved by induction on n , the rank of V , and the case $n = 1$ is just Theorem 5.2.

Let T and T' be maximal Q -valuation rings that contain R and R' respectively (Theorem 2.8). Then $T \cap K = V_P = T' \cap K$ where P is the minimal prime ideal in V . Theorem 5.2 implies that T and T' are conjugate in Q , say $qT'q^{-1} = T$ for some invertible element q in Q and $qJ(T')q^{-1} = J(T)$ follows.

The simple algebra $\overline{Q} = T/J(T)$, which is finite dimensional over its center Z , contains the \overline{Q} -valuation rings $R/J(T) = \overline{R}$ and $\overline{qR'q^{-1}} = \overline{qR'q^{-1}}/J(T)$. Let S be the separable closure of $\overline{K} = T \cap K/(J(T) \cap K)$ in Z .

Z is a field and $\overline{R} \cap Z$ and $\overline{qR'q^{-1}} \cap Z$ are two valuation rings in Z with $(\overline{R} \cap Z) \cap \overline{K} = (\overline{qR'q^{-1}} \cap Z) \cap \overline{K}$. Since T is a maximal Q -valuation ring we can apply Proposition 5.3 and conclude that there exists a regular element r in Q with $rTr^{-1} = T$ such that $\varphi_r(\overline{qR'q^{-1}} \cap Z) = \overline{R} \cap Z$ where φ_r is the mapping induced on $T/J(T)$ by the inner automorphism of Q corresponding to r . Hence, $\overline{rqR'q^{-1}r^{-1}} \cap Z = \overline{R} \cap Z$.

By induction, applied to $\overline{Q}, \overline{R}$ and $\overline{rqR'q^{-1}r^{-1}}$ there exists in $T \setminus J(T)$ an element a such that \overline{a} is invertible in \overline{Q} and $\overline{arqR'q^{-1}r^{-1}a^{-1}} = \overline{R}$. This implies that a is invertible in Q and $arqR'q^{-1}r^{-1}a^{-1} = R$.

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